Mather invariants in groups of piecewise-linear homeomorphisms

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Abstract

We describe the relation between two characterizations of conjugacy in groups of piecewise-linear homeomorphisms, discovered by Brin and Squier in [2] and Kassabov and Matucci in [5]. Thanks to the interplay between the techniques, we produce a simplified point of view of conjugacy that allows ua to easily recover centralizers and lends itself to generalization.

1 Introduction

We denote by $\mathrm{PL}_+(I)$ the group of orientation-preserving piecewise-linear homeomorphisms of the unit interval $I = [0, 1]$ with finitely many breakpoints. We will treat only the case of $\mathrm{PL}_+(I)$ even if all the results can be adapted to certain subgroups of $\mathrm{PL}_+(I)$ of homeomorphisms with certain requirements on the breakpoints and the slopes (for example, Thompson's group F and the groups $\mathrm{PL}_{S,G}(I)$ introduced in [5]). In particular, we will only work with functions that do not intersect the diagonal, except for the points 0 and 1.

In their work [2] Brin and Squier define an invariant under conjugacy for maps of $\mathrm{PL}_+(I)$ that do not intersect the diagonal. Their description is based on similar earlier work by Mather [6] for diffeomorphisms of the unit interval and allows the classification of centralizers and the detection of roots of elements. These techniques were originally introduced as an attempt to solve the conjugacy problem in Thompson's group F (which was then proved to be solvable by Guba and Sapir in [4]). Later on this approach was refined by Gill and Short in [3] and Belk and Matucci [1] to give another proof of the solution to the conjugacy problem in Thompson's group F. On the other hand, Kassabov and Matucci showed a solution to the simultaneous conjugacy problem in [5] by producing an algorithm to build all conjugators, if they exist. Similarly, these techniques can be used to obtain centralizers and roots as a byproduct.

The aim of this note is to show the connection between the techniques in [2] and [5] to characterize conjugacy in groups of piecewise-linear homeomorphisms. By defining a modified version of Brin and Squier's invariant and using a mixture of those points of view it is possible to produce a short proof of the description of conjugacy and centralizers in $\mathrm{PL}_+(I)$. In particular, the interplay between these two points of view lends itself to generalizations giving a tool to study larger class of groups of piecewise-linear homemorphisms.

This paper is organized as follows. In Section 2 we give a short account of a key algorithm in [5] (the *stair algorithm*) to build a particular conjugator g for two elements $y, z \in \mathrm{PL}_+(I)$. In Section 3 we define a conjugacy invariant (called *Mather invariant*) that essentially encodes the characterization of conjugacy in [2]

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for $\mathrm{PL}_+(I)$. In Section 4 we show to use the stair algorithm to simplify the proof of the the characterization of conjugacy of [2] using Mather invariants. In turn, in Section 5 we will show how Mather invariants allow us to shorten the arguments in [5] to classify centralizers of elements. We finish by briefly describing possible extensions of these tools.

2 The stair algorithm for functions in $\mathrm{PL}_+^<(I)$

In this Section we will discuss how to find a special conjugator $g \in \mathrm{PL}_+(I)$ for two functions $y, z \in \mathrm{PL}_+(I)$, if it exists. The idea will be to assume that such a conjugator g exists and obtain conditions that g must satisfy.

Definition 2.1. We denote by $\mathrm{PL}_+^<(I)$ the subset of $\mathrm{PL}_+(I)$ of all functions that lie below the diagonal, that is the maps $z \in \mathrm{PL}_+(I)$ such that $f(t) > t$ for all $t \in (0,1)$. Similarly, we define the subset $\mathrm{PL}_+^>(I)$ of functions that lie above the diagonal. A function $z \in \mathrm{PL}_+(I)$ is defined to be a *one-bump function* if either $z \in \mathrm{PL}_+^<(I)$ or $z \in \mathrm{PL}_+^>(I)$.

If $z \in \mathrm{PL}_+(I)$, we define *initial slope* and *final slope*, respectively, to be the numbers $z'(0)$ and $z'(1)$. It is clear that if two one-bump functions y and z are conjugate, their initial and final slope are the same. A more interesting fact is that a conjugator has to be linear in certain boxes around 0 and 1. This fact, together with the ability to identify the two functions step by step, allows us to build a conjugator.

Lemma 2.2 (Kassabov and Matucci, [5])**.** *Suppose $y, z \in \mathrm{PL}_+^<(I)$.*

1. *(**initial box**) Let $g \in \mathrm{PL}_+(I)$ be such that $g^{-1}yg = z$. Assume $y(t) = z(t) = ct$ for $t \in [0, \alpha]$ and $c < 1$. Then the graph of g is linear inside the box $[0,\alpha] \times [0,\alpha]$. A similar statement is true for a "final box"*
2. *(**identification trick**) Let $\alpha \in (0,1)$ be such that $y(t) = z(t)$ for $t \in [0,\alpha]$. Then there exists a $g \in \mathrm{PL}_+(I)$ such that $z(t) = g^{-1}yg(t)$ for $t \in [0, z^{-1}(\alpha)]$ and $g(t) = t$ in $[0,\alpha]$. The element g is uniquely defined up to the point $z(\alpha)$..*
3. *(**uniqueness of conjugators**) For any positive real number q there exists at most one $g \in \mathrm{PL}_+(I)$ such that $g^{-1}yg = z$ and $g'(0) = q$.*
4. *(**conjugator for powers**) Let $g \in \mathrm{PL}_+(I)$ and $n \in \mathbb{N}$. Then $g^{-1}yg = z$ if and only if $g^{-1}y^n g = z^n$.*

Proof. The proof of (1) is straightforward. To prove (2) we observe that, if such a g exists then, for $t \in [0, z^{-1}(\alpha)]$

$$y(g(t)) = g(z(t)) = z(t)$$

since $z(t) \le \alpha$. Thus $g(t) = y^{-1}z(t)$ for $t \in [0, z^{-1}(\alpha)]$. To prove that such a g exists, define

$$g(t) := \begin{cases} t & t \in [0,\alpha] \\ y^{-1}z(t) & t \in [\alpha, z^{-1}(\alpha)] \end{cases}$$

and extend it to I as a line from the point $(z^{-1}(\alpha), y^{-1}(\alpha))$ to $(1,1)$. To prove (3), assume that there exist two conjugators g_1, g_2 with initial slope q. Since $g_1^{-1}yg_1 = g_2^{-1}yg_2$ we have that $g := g_1 g_2^{-1}$ centralizes y and it has initial slope 1. Assume, by contradiction, that g is the identity on $[0,\alpha]$ for some α, but $g'(\alpha^+) \neq 1$. Since we have

$$y(g(t)) = g(y(t)) = y(t)$$

for $t \in [\alpha, y^{-1}(\alpha)]$, this implies that $g(t) = y^{-1}y(t) = t$ on $[\alpha, y^{-1}(\alpha)]$, which is a contradiction. To prove the last statement we observe that if $f := g^{-1}y^n g = z^n$, then f is centralized by both $g^{-1}yg$ and z. Since $g^{-1}yg$ and z have the same initial slope, then by (3) we have $g^{-1}yg = z$. $\square$

Part (1) of the previous Lemma tells us that any given conjugator g must be linear in two suitable boxes $[0,\alpha]^2$ and $[\beta,1]^2$, hence if we are given a point $(p, g(p))$ in any of those boxes (say the final one), we can draw the longest segment contained in $[\beta,1]^2$ passing through $(p, g(p))$ and $(1,1)$ and obtain the map g in that box. We are now going to build a candidate conjugator with a given initial slope.

Theorem 2.3 (Stair Algorithm, [5])**.** *Let $y, z \in \mathrm{PL}_+^<(I)$, let $[0,\alpha]^2$ be the initial linearity box and let $0 < q < 1$ be a real number. There is an $N \in \mathbb{N}$ such that the unique candidate conjugator with initial slope q is given by*

$$g(t) = y^{-N} g_0 z^N(t) \qquad \forall t \in [0, z^{-N}(\alpha)]$$

and linear otherwise, where g_0 is any map in $\mathrm{PL}_+(I)$ which is linear in the initial box and such that $g_0'(0) = q$. If g is indeed a conjugator of y and z, then $g = y^{-N} g_0 z^N$ on the interval I.

By "unique candidate conjugator" we mean a function g such that, if there exists a conjugator between y and z with initial slope q, then it must be equal to g. Hence we can test our candidate conjugator to verify if it is indeed a conjugator.

Proof. Let $[\beta,1]^2$ be the final box and N an integer big enough so that

$$\min\{z^{-N}(\alpha), y^{-N}(q\alpha)\} > \beta.$$

We will build a candidate conjugator g between y^N and z^N (if it exists) as a product of two functions g_0 and g_1. We note that the linearity boxes for y^N and z^N are still given by $[0,\alpha]^2$ and $[\beta,1]^2$. By Lemma 2.2 (1) g has to be linear on $[0,\alpha]$ and so we define an "approximate conjugator" g_0 by:

$$g_0(t) := qt \qquad t \in [0,\alpha]$$

and extend it to the whole I as a line through $(1,1)$. We then define $y_1 := g_0^{-1} y g_0$ and look for a conjugator g_1 of y_1^N and z^N, noticing that y_1^N and z^N coincide on $[0,\alpha]$. By Lemma 2.2(2), we define

$$g_1(t) := \begin{cases} t & t \in [\eta, \alpha] \\ y_1^{-1} z^N(t) & t \in [\alpha, z^{-N}(\alpha)] \end{cases}$$

and extend it to I as a line through $(1,1)$ so that $g_1^{-1} y_1^N g_1 = z^N$ on $[0, z^{-N}(\alpha)]$. Finally, build a function g such that $g(t) := g_0 g_1(t)$ for $t \in [0, z^{-N}(\alpha)]$ and extend it to I as a line through $(1,1)$ on $[z^{-N}(\alpha), 1]$. The map g is inside the final box at $t = z^{-N}(\alpha) > \beta$, in fact

$$g(z^{-N}(\alpha)) = g_0 g_0^{-1} y^{-N} g_0(\alpha) = y^{-N}(q\alpha) > \beta.$$

We observe that, by construction, g is a conjugator for y^N and z^N on $[0, z^{-N}(\alpha)]$, that is $g = g_0 g_1 = y^{-N} g_0 g_1 z^N$ on $[0, z^{-N}(\alpha)]$. Therefore

$$g(t) = y^{-N} g_0 g_1 z^N(t) = y^{-N} g_0 z^N(t) \qquad \forall t \in [0, z^{-N}(\alpha)]$$

since $g_1 z^N(t) = z^N(t)$ for $t \in [0, z^{-N}(\alpha)]$.

By Lemma 2.2(3), if there is a conjugator for y^N and z^N with initial slope q, it must be equal to g. So we just check if g conjugates y^N to z^N. Morever, Lemma 2.2(4) tells us that g is a conjugator for y^N and z^N if and only it is for y and z and so we are done. $\square$

Remark 2.4. We remark that the proof of the previous Theorem does not depend on the choice of g_0. The only requirements on g_0 are that it must be linear in the initial box and $g_0'(0) = q$.

3 Mather invariants for functions in $\mathrm{PL}_+^>(I)$

In this Section we will give an alternate description of Brin and Squier's conjugacy invariant in [2]. This reformulation was also used by Belk and Matucci in [1] to characterize conjugacy in Thompson's group F: however, their proof relies on special kinds of diagrams peculiar to F and cannot be generalized to other groups of homeomorphisms.

The Mather invariant of a map $z \in \mathrm{PL}_+^>(I)$ is defined by taking a suitable power of z that sends a neighborhood in the first linear segment of z to a neighborhood in the last linear segment of z.

Consider a one-bump function $z \in \mathrm{PL}_+^>(I)$, with slope m_0 at 0 and slope m_1 at 1. In a neighborhood of zero, z acts as multiplication by m_0; in particular, for any sufficiently small $t > 0$, the interval $[t, m_0 t]$ is a fundamental domain for the action of z.

If we make the identification $t \sim m_0 t$ in the interval $(0, \epsilon)$, for a sufficiently small $\epsilon > 0$, we obtain a circle C_0, with partial covering map $p_0 \colon (0, \epsilon) \to C_0$. Similarly, if we identify $(1 - t) \sim (1 - m_1 t)$ on the interval $(1 - \delta, 1)$, for a sufficiently small $\delta > 0$, we obtain a circle C_1, with partial covering map $p_1 \colon (1 - \delta, 1) \to C_1$.

If N is sufficiently large, then z^N will take some lift of C_0 to (ϵ', ϵ), for a sufficiently small $\epsilon' > 0$, and map it to the interval $(1 - \delta, 1)$. This induces a map $z^\infty \colon C_0 \to C_1$, making the following diagram commute:

$$\begin{tikzcd}
(\epsilon', \epsilon) \arrow[r, "z^N"] \arrow[d, "p_0"'] & (1-\delta, 1) \arrow[d, "p_1"] \\
C_0 \arrow[r, dashed, "z^\infty"] & C_1
\end{tikzcd}$$

The map z^∞ defined above is called the **Mather invariant** for z. We note that z^∞ does not depend on the specific value of N chosen. Any map z^m, for $m \geq N$, induces the same map z^∞. This is because z acts as the identity on C_1 by construction and z^m can be written as $z^{m-N}(z^N(t))$, with $z^N(t) \in (1 - \delta, 1)$. If $k > 0$, then the map $t \mapsto kt$ on $(0, \epsilon)$ induces a "rotation" rot_k of C_0. In particular, if we use the coordinate $\theta = \log t$ on C_0, then

$$\mathrm{rot}_k(\theta) \;=\; \theta + \log k$$

so rot_k is an actual rotation. In the next Section we will give a characterization of conjugacy for one-bump functions by means of Mather invariants.

4 Equivalence of the two points of view

In this Section we will show the relation between the stair algorithm and the definition of Mather invariant. This will provide an alternative proof of Brin and Squier's conjugacy invariant.

Theorem 4.1 (Brin and Squier, [2])**.** *Let $y, z \in \mathrm{PL}_+^>(I)$ be one-bump functions with $y'(0) = z'(0)$ and $y'(1) = z'(1)$, and let $y^\infty, z^\infty \colon C_0 \to C_1$ be the corresponding*

Mather invariants. Then y and z are conjugate if and only if y^∞ and z^∞ differ by rotations of the domain and range circles :

$$\begin{tikzcd}
C_0 \arrow[r, "y^\infty"] \arrow[d, "\mathrm{rot}_k"'] & C_1 \arrow[d, "\mathrm{rot}_\ell"] \\
C_0 \arrow[r, "z^\infty"'] & C_1
\end{tikzcd}$$

Proof. Suppose that $z = g^{-1}yg$ for some $g \in \mathrm{PL}_+(I)$. Then the following diagram commutes, where $k = g'(0)$ and $\ell = g'(1)$:

$$\begin{tikzcd}
& (\epsilon', \epsilon) \arrow[rr, "y^N"] \arrow[dd, "p_0"] & & (1-\delta, 1) \arrow[dd, "p_1"] \\
(\epsilon', \epsilon) \arrow[ur, "g"] \arrow[rr, "z^N"] \arrow[dd, "p_0"'] & & (1-\delta, 1) \arrow[ur, "g"] \arrow[dd, "p_1"] & \\
& C_0 \arrow[rr, "y^\infty"] & & C_1 \\
C_0 \arrow[ur, "\mathrm{rot}_k"'] \arrow[rr, "z^\infty"'] & & C_1 \arrow[ur, "\mathrm{rot}_\ell"'] &
\end{tikzcd}$$

To show the converse, we will prove that the two inverse maps y^{-1} and z^{-1} are conjugate. By assumption the maps y^∞ and z^∞ differ by rotations of the domain and the range circles, hence

$$\mathrm{rot}_\ell \, z^\infty = y^\infty \, \mathrm{rot}_k \, .$$

Up to passing to inverses and using the fact that $\mathrm{rot}_k^{-1} = \mathrm{rot}_{\frac{1}{k}}$, we can assume that $0 < k < 1$. Following the notation of Section 2, let $[0, \alpha]$ and $[\beta, 1]$ be the initial and the final boxes for y^{-1} and z^{-1}. Let g_0 be a map in $\mathrm{PL}_+(I)$ which is linear in the initial box and such that $g_0'(0) = \ell$. By Theorem 2.3 we have that $g := y^N g_0 z^{-N}$ is the unique conjugator for y^{-1} and z^{-1} on the interval $[0, z^N(\alpha)]$. We need to verify that g is a conjugator for y^{-1} and z^{-1} on $[z^N(\alpha), 1]$. Since $(z^N(\alpha), g(z^N(\alpha))$ is inside the final box, we need to verify that g is linear on $[g^{-1}(\beta), 1]$. Assume that there is an interval of the type $[p, z(p)] \subseteq [g^{-1}(\beta), 1]$ where g is not linear. By definition of g we have $gz^N = y^N g_0$ and we can build a diagram analogue to the one of "only if" part of this Theorem

$$\begin{tikzcd}
& (\epsilon', \epsilon) \arrow[rr, "y^N"] \arrow[dd, "p_0"] & & (1-\delta, 1) \arrow[dd, "p_1"] \\
(\epsilon', \epsilon) \arrow[ur, "g_0"] \arrow[rr, "z^N"] \arrow[dd, "p_0"'] & & (1-\delta, 1) \arrow[ur, "g"] \arrow[dd, "p_1"] & \\
& C_0 \arrow[rr, "y^\infty"] & & C_1 \\
C_0 \arrow[ur, "\mathrm{rot}_k"'] \arrow[rr, "z^\infty"'] & & C_1 \arrow[ur, "g_{\mathrm{ind}}"'] &
\end{tikzcd}$$

where the map g_{ind} is built passing the interval $[p, z(p)]$ to quotients. Since g is not linear on $[p, z(p)]$ then the induced map g_{ind} is not a rotation, and this is not possible since

$$\text{rot}_\ell\, z^\infty = y^\infty\, \text{rot}_k = g_{\text{ind}} z^\infty$$

and so $g_{\text{ind}} = \text{rot}_\ell$, by cancellation. Hence g is linear on $[g^{-1}(\beta), 1]$ with slope ℓ and it is straightforward to verify that g conjugates y^{-1} to z^{-1} inside the box $[\beta, 1]^2$. Thus $gy^{-1}g = z^{-1}$ and we are done. $\square$

Remark 4.2. The previous proof shows that two functions y, z are conjugate if and only if the explicit conjugator $g = y^N g_0 z^{-N}$ is linear in the final linearity box and this happens if and only if the two Mather invariants differ by rotations of the domain and the the range circles. The Mather invariant thus gives the "obstruction" to finishing the stair algorithm at 1.

Remark 4.3. We stress that the definition of Mather invariant and the construction of the stair algorithm do not really depend upon the set of breakpoints and slopes of the maps y and z. With little work, the two constructions and their equivalence can be extended to generalized Thompson's groups and, more generally, to the subgroups $\text{PL}_{S,G}(I)$ introduced in [5].

5 Applications: centralizers and generalizations

Given a map $f : S^1 \to S^1$, a *lift* of f is a map $F : \mathbb{R} \to \mathbb{R}$ such that $F(t+1) = F(t)+1$ for all $t \in \mathbb{R}$ and F induces f when passing the domain and the range to quotients via the relation $\alpha \sim \alpha + 1$. Given a lift, we talk about a *maximal V-interval* to refer to an interval $[a, b]$ such that F is linear with slope V on $[a, b]$ and a, b are breakpoints for F. We will give a short proof of the following well known result.

Theorem 5.1. *Let $z \in PL_+^>(I)$. Then the centralizer subgroup $C_{PL_+(I)}(z) = \{g \in PL_+(I) \mid gz = zg\}$ is isomorphic to the infinite cyclic group.*

Proof. Define the following group homomorphism:

$$\begin{array}{rccc} \varphi_z : & C_{\text{PL}_+(I)}(z) & \longrightarrow & (\mathbb{R}, +) \\ & g & \longmapsto & \log g'(0). \end{array}$$

Lemma 2.2(4) implies that φ_z is injective. By Theorem 4.1 any function g centralizing z induces two rotations $\text{rot}_\ell, \text{rot}_k$ such that

$$\text{rot}_\ell\, z^\infty = z^\infty\, \text{rot}_k$$

where $k = g'(0)$ and $\ell = g'(1)$. Observe that $R_\ell(t) = t + \log \ell$ and $R_k(t) = t + \log k$ are lifts of the two rotations $\text{rot}_\ell, \text{rot}_k$. Choose a lift $Z : \mathbb{R} \to \mathbb{R}$ of z^∞. The previous equality implies:

$$Z(t) + \log \ell = R_\ell(Z(t)) = Z(R_k(t)) = Z(t + \log k)$$

which means that the graph of Z can be shifted "diagonally" onto itself. The map Z is piecewise-linear and, for any positive number r, has finitely many breakpoints on the interval $[-r, r]$. Hence Z has only finitely many maximal $Z'(0)$-intervals that are contained in $[-r, r]$ and so there is only a discrete set of shifts (that is, values of $\log k = \varphi_z(g)$) which maps the graph of Z onto itself, unless Z is a line (and this is impossible since $z'(0) < z'(1)$ and so z^∞ must have breakpoints). Thus the image of φ_z must be a discrete subgroup of $(\mathbb{R}, +)$ and so, by a standard fact, it is isomorphic to $\mathbb{Z}$. $\square$

The Mather invariant approach is also interesting because it lends itself to generalizations. Let $\mathrm{PL}_{\mathrm{dis}}(\mathbb{R})$ the group of all orientation-preserving piecewise-linear homeomorphisms of the real line with a discrete set of breakpoints and let EP be the subgroup of $\mathrm{PL}_{\mathrm{dis}}(\mathbb{R})$ of the functions that are "eventually periodic at infinity", that is functions $f \in \mathrm{PL}_{\mathrm{dis}}(\mathbb{R})$ such that there exist numbers L_f, R_f so that $f(t-1) = f(t) - 1$ for $t < L_f$ and $f(t+1) = f(t) + 1$ for $t > R_f$. It is easy to define the subset $\mathrm{EP}^{>}$ and Mather invariant for functions in $\mathrm{EP}^{>}$: we just mod out the intervals $(-\infty, L_f)$ and (R_f, ∞) by the relation $t \sim f(t)$ and then take a power of f high enough so that $(f^{-1}(L_f), L_f)$ gets carried to a subset of (R_f, ∞). Similarly, one can partially extend the stair algorithm to build conjugators. It is thus interesting to see how much of these techniques can be extended to overgroups containing $\mathrm{PL}_+(I)$ to compute centralizers and, possibly, to study the conjugacy problem.

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